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Direct Gauging of the Poincaré Group

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Realization of the Poincaré group as a subgroup of $GL(5, R)$ that maps an affine set into itself is shown to lead to a well-defined minimal replacement operator when the Poincaré group is allowed to act locally. The minimal replacement operator is obtained by direct application of the Yang–Mills procedure without the explicit introduction of fiber bundle techniques. Its application gives rise to compensating 1-forms W^α, $1 \leq \alpha \leq 6$, for the local action of the Lorentz group $L(4, R)$, and to compensating 1-forms ϕ^k, $1 \leq k \leq 4$, for the translation group $T(4)$. When applied to the basis 1-forms dx^i of Minkowski space, distortion 1-forms B^k result that define a canonical anholonomic coframe that contains both the $T(4)$ and the $L(4, R)$ compensating fields. When the canonical coframe is considered as a differential system on M_4, it gives rise to gauge curvature expressions and Cartan torsion, but the latter has important differences from that usually encountered in the associated literature in view of the inclusion of the compensating fields for $L(4, R)$. The standard Yang–Mills minimal coupling construct is used to obtain a total Lagrangian. This leads to a system of field equations for the matter fields, the $T(4)$ compensating fields, and the $L(4, R)$ compensating fields. Part of the current that drives the $T(4)$ compensating fields is the 3-form of gauge momentum energy that obtains directly from the momentum–energy tensor of the matter fields on M_4 under minimal replacement. Introduction of the Cartan torsion in the free-field Lagrangian is shown to lead to a direct spin decoupling in the sense that the gauge momentum energy (orbital) contribution of the matter fields to the spin current is eliminated. Explicit conservation laws for total momentum energy current and total spin current are obtained.

1. INTRODUCTION

Gauge theories of gravity arise from the expectation that gravitational effects can be incorporated into a Poincaré invariant theory by allowing the Poincaré group to act locally (i.e., by breaking the global homogeneous action of the Poincaré group). Since $P_{10} = L(4, R) \triangleright T(4)$, where $L(4, R)$

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is the Lorentz group, $T(4)$ is the Abelian translation group, and $\triangleright$ denotes semidirect product, local action of $L(4, R)$ would model spin-gravitational interactions, while local action of $T(4)$ would account for the momentum-energy contributions to gravitation in the Einstein sense.

The current situation is not this clear cut, however. Because P_{10} is not semisimple, the Abelian ideal of the Lie algebra of P_{10}, which is generated by the Lie subalgebra of $T(4)$, gives certain difficulties. Most current works (Hehl et al., 1976; Hehl, 1980; Nitsch, 1980; Kikkawa et al., 1983; and references therein) start with the geometry. The underlying space is taken to be a four-dimensional Riemann-Cartan space U_4 carrying curvature and torsion. The primary aspect of these theories is a system of fundamental tetrad frames and coframes. The coframes $\mathbf{e}^\alpha$ are then identified with the gauge potential 1-forms that compensate for the local action of $T(4)$. Thus, if Ψ (indices suppressed) denotes the matter fields, minimal replacement becomes

$$\mathcal{M}(\partial_i \Psi) = e^i_\alpha D_i \Psi, \tag{1}$$

and

$$\mathcal{M}\big(L(x^i, \Psi, \partial_i \Psi)\mu\big) = \det(e^\beta_j) L(x^i, \Psi, e^i_\alpha D_i \Psi)\mu \tag{2}$$

where

$$\mu = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$$

is the volume element and $\mathcal{M}$ denotes the operation of minimal replacement.

Although the results obtained in this way stem from a number of plausible physical arguments, there would still seem to be certain difficulties. These arise from the fact that the geometry is put in first and then the physics is overlayed by means of reasonable analogies with classical Yang-Mills gauge theory. Particular conceptual difficulties reside in the relation represented by (2). Since the $\mathbf{e}^\alpha$ are identified with the compensating 1-forms for $T(4)$, it is evident that the minimal replacement construct defined by (2) ignores the contribution from the $L(4, R)$ component in compensating for the volumetric deformation [i.e., $\mathcal{M}(\mu) = \det(\mathbf{e}^\beta)\mu$]. This same situation is also in evidence in the effect of minimal replacement applied to the line element:

$$\mathcal{M}(h_{ij}\, dx^i\, dx^j) = g_{ij}\, dx^i\, dx^j \tag{3}$$

where

$$h_{ij} = \mathrm{diag}(-1, -1, -1, 1) \tag{4}$$

is the metric on Minkowski space M_4 and

$$g_{ij} = e^\alpha_i h_{\alpha\beta} e^\beta_j$$

Thus, since $\det(\mathbf{e}^\alpha)=(-g)^{1/2}$, so that (2) is consistent with the assumption that U_4 carries a metric structure given by the above relations, there is still a clear absence of effects of the local action of the $L(4, R)$ component in this metric structure.

Conceptually, a simpler approach would be to start with a Poincaré invariant theory of matter fields Ψ on Minkowski space, so that the physics is well established from the start. This would put the physics in first. Allowance for local action of the Poincaré group would then couple the matter fields to the compensating fields for the local action of P_{10} provided a consistent procedure is available for dealing with groups whose Lie algebra contains an Abelian ideal. A previous paper (Edelen, 1984) has provided such a procedure through study of operator-valued Lie connections. This procedure was used to gauge the Poincaré group, and hence it is explicitly applicable here. Unfortunately, the constructs of this paper are rather abstract because the problem of gauging an arbitrary Lie group was considered. There is, however, a simple and direct manner of gauging the Poincaré group that eliminates the complications of operator-valued connections, but which agrees exactly with the results obtained through the more abstract analysis. This direct approach is the subject of this paper.

2. MATRIX REPRESENTATION AND DIRECT MINIMAL REPLACEMENT

The simplest way of obtaining a correct minimal replacement for the Poincaré group is through realization of the Poincaré group as a matrix Lie group. To this end, we consider the affine set in V_5 consisting of all column matrices of the form

$$\hat{\mathbf{x}}=[x^1, x^2, x^3, x^4, 1]^T \tag{5}$$

where $\mathbf{x}=[x^1, x^2, x^3, x^4]^T$ is any position column matrix in M_4. If $\mathbf{L}$ is a Lorentz transformation matrix and $\mathbf{t}=[t^1, t^2, t^3, t^4]^T$ is a translation generator, then the Poincaré group may be realized as a matrix subgroup of $GL(5, R)$ consisting of all 5-by-5 matrices of the form

$$\mathbf{M}=\begin{pmatrix} \mathbf{L} & \mathbf{t} \\ [0] & 1 \end{pmatrix} \tag{6}$$

that is,

$${}'\hat{\mathbf{x}}=\mathbf{M}\hat{\mathbf{x}}=\begin{Bmatrix} \mathbf{Lx}+\mathbf{t} \\ 1 \end{Bmatrix} \tag{7}$$

Let $\{\mathbf{1}_\alpha \mid 1\leq\alpha\leq 6\}$ be a basis for the matrix Lie algebra of $L(4, R)$, and let $\{\mathbf{e}_i \mid 1\leq i\leq 4\}$ be a system of generators for $T(4)$. Here, we have taken canonical group coordinates in a neighborhood of the identity of P_{10} to be $\{u^a \mid 1 \le a \le 10\} = \{u^\alpha, u^{6+i} \mid 1 \le \alpha \le 6, 1 \le i \le 4\}$. If $\{W^a \mid 1 \le a \le 10\}$ are 1-forms on M_4, then we write $\{W^a\} = \{W^\alpha, \phi^i \mid 1 \le \alpha \le 6, 1 \le i \le 4\}$, where we have set $W^{6+i} = \phi^i$. Since P_{10} has been realized as a matrix Lie group on V_5, we may use the standard Yang-Mills construction of a Lie-algebra-valued connection matrix for P_{10} (Yang, 1975; Drechrler and Mayer, 1977; Rund, 1982):

$$\hat{\mathbf{\Gamma}} = \begin{pmatrix} W^\alpha \mathbf{1}_\alpha & \phi^i \mathbf{e}_i \\ [0] & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma} & \boldsymbol{\omega} \\ [0] & 0 \end{pmatrix} \tag{8}$$

It is clear from (8) that $\mathbf{\Gamma}$ may be viewed as a connection matrix for the $L(4, R)$ component, while $\boldsymbol{\omega}$ is a connection matrix for the local action of $T(4)$. The standard Yang-Mills construct thus gives the gauge covariant exterior derivative

$$D\hat{\mathbf{x}} = d\hat{\mathbf{x}} + \hat{\mathbf{\Gamma}}\hat{\mathbf{x}} = \begin{Bmatrix} d\mathbf{x} + \mathbf{\Gamma}\mathbf{x} + \boldsymbol{\omega} \\ 0 \end{Bmatrix} \tag{9}$$

When this is written out in index notation, we have

$$Dx^i = dx^i + W_j^\alpha 1_{\alpha k}^{\ i} x^k dx^j + \phi_j^i dx^j \tag{10}$$

and hence

$$Dx^i = B_j^i \, dx^j \tag{11}$$

where

$$B_j^i = \delta_j^i + W_j^\alpha 1_{\alpha k}^{\ i} x^k + \phi_j^i \tag{12}$$

These formulas show that minimal replacement is well defined for local action of the Poincaré group on the elements $\{dx^i\}$:

$$\mathcal{M}(dx^i) = Dx^i := B^i = B_j^i \, dx^j \tag{13}$$

The quantities $\{B^i \mid 1 \leqslant i \leqslant 4\}$ are referred to as the distortion 1-forms for the Poincaré group, in analogy with defect dynamics (Kadić and Edelen, 1983) that arises from gauging $SO(3) \rhd T(3)$.

Examination of (10) shows that both the translation and the $L(4, R)$ compensating fields enter into the Poincaré distortion 1-forms. This is in strong contrast to previous gauge theories for the Poincaré group noted in the first section. The situation becomes even more evident when we compute what happens to the line element of M_4 under minimal replacement:

$$dS^2 = \mathcal{M}(h_{ij} \, dx^i \, dx^j) = g_{ij} \, dx^i \, dx^j \tag{14}$$

where

$$g_{ij} = B_i^k h_{km} B_j^m = g_{ji} \tag{15}$$

may be viewed as the components to the metric tensor on a space U_4 that results from M_4 under the operation of minimal replacement. Thus if we set

$$B = \det(B^i_j), \qquad g = \det(g_{ij}) \tag{16}$$

then

$$B = (-g)^{1/2} \tag{17}$$

and minimal replacement on the volume element of M_4 gives

$$\mathcal{M}(\mu) = B\mu = (-g)^{1/2}\mu \tag{18}$$

In like manner, if we set

$$\hat{\boldsymbol{\theta}} = d\hat{\boldsymbol{\Gamma}} + \hat{\boldsymbol{\Gamma}} \wedge \hat{\boldsymbol{\Gamma}} \tag{19}$$

then (8) gives

$$\hat{\boldsymbol{\theta}} = \begin{pmatrix} \theta^\alpha \mathbf{1}_\alpha & \Omega^i \mathbf{e}_i \\ [0] & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\theta} & \boldsymbol{\Omega} \\ [0] & 0 \end{pmatrix} \tag{20}$$

If we use capital C's to denote the constants of structure of P_{10}, direct computation gives

$$\theta^\alpha = dW^\alpha + \tfrac{1}{2} C_\beta{}^\alpha{}_\gamma W^\beta \wedge W^\gamma \tag{21}$$

for the $L(4, R)$ curvature, and

$$\Omega^i = d\phi^i + C_\alpha{}^i{}_j W^\alpha \wedge \phi^j \tag{22}$$

for the $T(4)$ part of the curvature. Here, we have used the fact that the structure constants for P_{10} satisfy

$$C_\alpha{}^i{}_\beta = C_i{}^j{}_k = C_i{}^\alpha{}_j = 0 \tag{23}$$

Curvature quantities arise directly through the relations

$$DD\hat{\mathbf{x}} = \hat{\boldsymbol{\theta}}\hat{\mathbf{x}} = [(\boldsymbol{\theta}\mathbf{x} + \boldsymbol{\Omega})^T, 0]^T \tag{24}$$

Let us first note that (24) may be rewritten as

$$DB^i = DDx^i = \theta^\alpha 1_{\alpha k}^{\ i} x^k + \Omega^i \tag{25}$$

while the torsion associated with the differential system generated by the 1-forms $\{B^i \mid 1 \le i \le 4\}$ (see Edelen, 1985) is defined by

$$DB^i = \Sigma^i \tag{26}$$

Thus, (25) and (26) give the following evaluation for the torsion 2-forms:

$$\Sigma^i = \theta^\alpha 1_{\alpha k}^{\ i} x^k + \Omega^i \tag{27}$$

It is then a simple matter to see from (25) through (27) that the coframe

$\{B^i \mid 1 \le i \le 4\}$ is anholonomic whenever the torsion 2-forms do not vanish identically. Further, in contrast with previous theories, the torsion 2-forms have contributions from the compensating fields of both the $T(4)$ and the $L(4, R)$ components. This result have obvious and important ramifications. In particular, minimial replacement for P_{10} may be viewed as carrying M_4 to a new space U_4 with metric tensor g_{ij} and fundamental coframe $\{B^i\}$, in which case U_4 has both nontrivial curvature and torsion.

Direct use of classical Yang-Mills theory in the context of the matrix representation of P_{10} shows that we have the following gauge transformations for the connection and curvature matrices:

$$'\hat{\mathbf{\Gamma}} = \mathbf{M}\hat{\mathbf{\Gamma}}\mathbf{M}^{-1} - d\mathbf{M}\,\mathbf{M}^{-1}, \qquad '\hat{\boldsymbol{\theta}} = \mathbf{M}\hat{\boldsymbol{\theta}}\mathbf{M}^{-1} \tag{28}$$

When (6), (8), and (20) are substituted into (28), we obtain the following explicit transformation laws:

$$'\mathbf{\Gamma} = \mathbf{L}\mathbf{\Gamma}\mathbf{L}^{-1} - d\mathbf{L}\,\mathbf{L}^{-1} \tag{29}$$

$$'\boldsymbol{\omega} = \mathbf{L}\boldsymbol{\omega} - {}'\mathbf{\Gamma}\mathbf{t} - d\mathbf{t} \tag{30}$$

$$'\boldsymbol{\theta} = \mathbf{L}\boldsymbol{\theta}\mathbf{L}^{-1} \tag{31}$$

$$'\mathbf{\Omega} = \mathbf{L}\mathbf{\Omega} - {}'\boldsymbol{\theta}\mathbf{t} \tag{32}$$

To round out these transformation formulas, it is easily seen that

$$'\mathbf{B} = {}'(D\mathbf{x}) = \mathbf{L}D\mathbf{x} = \mathbf{L}\mathbf{B} \tag{33}$$

Simple calculation then shows that it is possible to choose a local gauge transformation [i.e., choose $\mathbf{L}(x^i)$ and $\mathbf{t}(x^i)$] that annihilates $\mathbf{\Gamma}$ and $\boldsymbol{\omega}$ if and only if all of the curvature quantities vanish identically on M_4. Thus the compensating fields W^α and ϕ^k can be transformed away by the local action of an element of P_{10} if and only if all gauge curvature quantities vanish throughout M_4.

An inspection of (31) shows that the quantity

$$\operatorname{trace}(\boldsymbol{\theta} \wedge \boldsymbol{\theta})$$

is a P_{10}-covariant 4-form. We thus have the P_{10}-invariant scalar densities

$$\begin{aligned} \beta_1 &= \theta^\alpha_{ij} C_{\alpha\beta} \theta^\beta_{km} g^{ik} g^{jm} B \\ \beta_2 &= \theta^\alpha_{ij} C_{\alpha\beta} \theta^\beta_{km} e^{ijkm} \end{aligned} \tag{34}$$

where $C_{\alpha\beta}$ are the components of the Cartan-Killing metric on $L(4, R)$. Here, we have used the obvious notation

$$2\theta^\alpha = \theta^\alpha_{ij}\, dx^i \wedge dx^j, \qquad 2\Omega^i = \Omega^i_{jk}\, dx^j \wedge dx^k \tag{35}$$

etc. A direct inspection of (30) and (32) shows that we cannot construct a

P_{10}-invariant 4-form from the $T(4)$ part of the curvature $\mathbf{\Omega}$ because of the arbitrary translation matrix. It is, however, easily seen from the above transformation equations that the Cartan torsion has the transformation law

$$'\mathbf{\Sigma} = \mathbf{L}\mathbf{\Sigma}$$

and we accordingly have the P_{10}-invariant scalar density

$$\begin{aligned} \alpha_1 &= \Sigma^i_{jk} h_{im} \Sigma^m_{rs} g^{jr} g^{ks} B \\ \alpha_2 &= \Sigma^i_{jk} h_{im} \Sigma^m_{rs} e^{jkrs} \end{aligned} \tag{36}$$

The scalar density P_{10}-invariants will be of obvious use later.

3. MINIMAL REPLACEMENT FOR THE MATTER FIELDS

We now have to obtain the minimal replacement construct for the matter fields. In order to keep things straight, it is useful to introduce a collective index $\{A\}$ in order to identify the various components of the matter field. We will therefore write $\{\Psi^A\}$, and the summation convention is assumed to hold for this collective index. Let $\{\Delta u^a\}$ denote a point in the infinitesimal neighborhood of the identity in the group space of P_{10}. Since the matter fields have well-defined transformation properties under the global action of P_{10}, we have

$$\Delta P_{10}: \Psi^A \to \Psi^A + \Delta u^\alpha (f_\alpha \Psi)^A + \Delta u^{6+i} (f_i \Psi)^A + o(\Delta) \tag{37}$$

where $\{f_\alpha \mid 1 \le \alpha \le 6\}$ are the $L(4, R)$ generators acting on the matter fields, while $\{f_i \mid 1 \le i \le 4\}$ are the $T(4)$ generators. For most matter fields considered in the literature, $f_i = 0$. We retain them in the formalism, however, in view of possible future needs. A comparison of (37) with the formalism used in (Edelen, 1984) shows that the quantities in (37) that multiply the Δu's serve to evaluate the Lie derivatives of the matter fields with respect to the infinitesimal generating vectors of P_{10} on kinematic space with coordinates $\{x^i, \Psi^A\}$. We thus have the gauge covariant derivative

$$D\Psi^A = d\Psi^A + W^\alpha (f_\alpha \Psi)^A + \phi^i (f_i \Psi)^A \tag{38}$$

Now,

$$0 = d\Psi^A - \partial_i \Psi^A \, dx^i$$

and hence minimal replacement gives

$$0 = D\Psi^A - \mathcal{M}(\partial_i \Psi^A) Dx^i$$

Thus if we set

$$y_i^A = \mathcal{M}(\partial_i \Psi^A) \tag{39}$$

and use (11) and (38), we obtain the minimal replacement

$$y_i^A B_j^i = \partial_j \Psi^A + W_j^\alpha (f_\alpha \Psi)^A + \phi_j^k (f_k \Psi)^A \tag{40}$$

now explicitly assume the regularity condition

$$B \neq 0 \tag{41}$$

Under satisfaction of (41), we have

$$b_k^i B_j^k = B_k^i b_j^k = \delta_j^i \tag{42}$$

and hence (40) gives the explicit evaluation

$$\mathcal{M}(\partial_i \Psi^A) = y_i^A = b_i^j \{\partial_j \Psi^A + W_j^\alpha (f_\alpha \Psi)^A + \phi_j^k (f_k \Psi)^A\} \tag{43}$$

This result seems unduly complicated on first examination. However, when all of the compensating fields W^α and ϕ^i all vanish, both **B** and **b** become the identity matrix and (41) reduces to $\mathcal{M}(\partial_i \Psi^A) = \partial_i \Psi^A$, as indeed it must. Further, the quantities within the curly brackets are the components of the gauge covariant derivatives of the matter fields, while the left multiplication by **b** accounts for the presence of local Poincaré transformations of the independent variables. Thus, (41) is exactly what is required in view of the fact that P_{10} acts on both the independent variables and the matter fields simultaneously.

4. MINIMAL COUPLING

The physics of the matter fields is described by a Poincaré invariant action 4-form $L\mu$ on Minkowski space. Under local action of P_{10}, the action 4-form becomes

$$\mathcal{M}(L(x^i, \Psi^A, \partial_i \Psi^A)\mu) = L(x^i, \Psi^A, y_i^A) B\mu \tag{44}$$

The modified Lagrangian, LB, does not depend on any of the derivatives of the compensating fields W^α or ϕ^i. This is remedied by the now famous Yang–Mills minimal coupling construct; LB is replaced by $LB + V$, where V is an invariant scalar density that is formed out of the P_{10} curvature quantities $\boldsymbol{\theta}^\alpha$ and Ω^i, respectively. We accordingly have the total Lagrangian

$$\bar{L} = L(x^i, \Psi^A, y_i^A) B + V(x^i, \theta_{ij}^\alpha, \Omega_{ij}^k, W_i^\alpha, \phi_i^k) \tag{45}$$

The end of the second section showed that the quantities β are P_{10}-invariant scalar densities formed from the $L(4, R)$ curvature coefficients and are quadratic in the latter. On the other hand, we saw that it was impossible to form a similar P_{10}-invariant scalar density from the curvature coefficients associated with $T(4)$. Rather, it was necessary that we construct the requisite scalar densities α from the coefficients of the Cartan torsion

Σ. It would thus, perhaps, be better to write

$$\bar{V}(x^i, \theta^\alpha_{ij}, \Sigma^k_{ij}) \text{ instead of } V(x^j, \theta^\alpha_{ij}, \Omega^k_{ij}) \tag{46}$$

Here, we recall for later purposes that the components of the Cartan torsion are given by

$$\Sigma^k_{ij} = \theta^\alpha_{ij} 1_{\alpha m}{}^k x^m + \Omega^k_{ij} \tag{47}$$

The combined action functional for the P_{10} gauge field theory of matter fields is thus given by

$$I[\Psi^A, W^\alpha_i, \phi^k_i] = \int_{M_4} (LB + V)\mu \tag{48}$$

5. FIELD EQUATIONS

The field equations for the theory obtained directly by rendering the action integral (48) stationary with respect to the field variables

$$\Psi^A, \qquad W^\alpha_i, \qquad \text{and} \qquad \phi^k_i$$

Variation with respect to these field variables induces variations in the subsidiary quantities

$$B, \qquad y^A_i, \qquad \theta^\alpha_{ij}, \qquad \Omega^k_{ij}$$

These are readily computed through use of (12), (16), (21), (22), and (40). The only slightly difficult part arises with the y's, and here (40) rather than (43) is definitely easier to use.

The total Lagrangian, $\bar{L}$, is given by $\bar{L} = LB + V$, where L does not depend on the θ's and Ω's, and V does not depend on the Ψ's and the y's. The following notation will therefore be useful:

$$G^{ij}_\alpha = -G^{ji}_\alpha = \partial V / \partial \theta^\alpha_{ij}, \qquad G^{ij}_k = -G^{ji}_k = \partial V / \partial \Omega^k_{ij} \tag{49}$$

$$S^i_\alpha = (\partial V / \partial W^\alpha_i)|_{\theta,\Omega}, \qquad S^i_k = (\partial V / \partial \phi^k_i)|_{\theta,\Omega} \tag{50}$$

where the notation in (50) is used to signify evaluation of the derivatives at constant θ's and Ω's, and

$$L^i_A = \partial L / \partial y^A_i \tag{51}$$

These latter quantities enter into what we term the gauge momentum energy complex of the matter fields,

$$T^i_j = L^i_A y^A_j - \delta^i_j L, \tag{52}$$

since they obtain by direct minimal replacement applied to the momentum energy complex of the matter fields.

Direct calculation shows that the field equations for the matter fields with P_{10} gauge field interactions are given by

$$\partial_j\{Bb^j_i L^i_A\} - B\,\partial L/\partial\Psi^A = Bb^j_i L^i_E\{W^\alpha_j\,\partial(f_\alpha\Psi)^E/\partial\Psi^A + \phi^k_j\,\partial(f_k\Psi)^E/\partial\Psi^A\} \tag{53}$$

Stationarization of the action functional with respect to the ϕ's gives the field equations for the translation compensating fields

$$\partial_i G^{ij}_k + C_k{}^m{}_\alpha W^\alpha_i G^{ij}_m = \tfrac{1}{2}B\{-T^m_k + L^m_A(f_k\Psi)^A\}b^j_m + \tfrac{1}{2}S^j_k \tag{54}$$

The $L(4, R)$ compensating field equations are

$$\begin{aligned}\partial_i G^{ij}_\alpha + C_\alpha{}^\beta{}_\gamma W^\gamma_i G^{ij}_\beta + C_\alpha{}^m{}_k \phi^k_i G^{ij}_m \\ = \tfrac{1}{2}B\{-T^m_k 1_\alpha{}^k{}_r x^r + L^m_A(f_\alpha\Psi)^A\}b^j_m + \tfrac{1}{2}S^j_\alpha\end{aligned} \tag{55}$$

Fields of elementary matter are customarily envisioned as representations of the isotropy subgroup of P_{10}, namely, $L(4, R)$. Restricting attention to such fields gives

$$(f_\alpha\Psi)^A = M_{\alpha B}^{\ A}\Psi^B, \qquad (f_i\Psi)^A = 0 \tag{56}$$

where the M's are the explicit representation matrices induced by the infinitesimal transformations of $L(4, R)$. When the evaluations (56) are used, the field equations become

$$\partial_j\{Bb^j_i L^i_A\} - W^\alpha_j M_{\alpha A}^{\ B}\{Bb^j_i L^i_\beta\} = B\,\partial L/\partial\Psi^A \tag{57}$$

$$2\{\partial_i G^{ij}_k - W^\alpha_i C_\alpha{}^m{}_k G_m{}^{ij}\} = -T^m_k Bb^j_m + S^j_k \tag{58}$$

$$2\{\partial_i G^{ij}_\alpha - W^\gamma_i C_\gamma{}^\beta{}_\alpha G^{ij}_\beta\} = S^j_\alpha - \{T^m_k 1_\alpha{}^k{}_r x^r - L^m_A M_{\alpha E}^{\ A}\Psi^E\}Bb^j_m - \phi^k_i C_\alpha{}^m{}_k G^{ij}_m \tag{59}$$

The quantities $\mu_i = \partial_i \lrcorner \mu$, $1 \le i \le 4$, form a conjugate basis for 3-forms on M_4. Thus since

$$\mathcal{M}(\partial_i) = b^j_i\,\partial_j, \qquad 1 \le i \le 4 \tag{60}$$

are the elements of the basis dual to the distortion 1-forms B^i, and $\mathcal{M}(\mu) = B\mu$, we have

$$\mathcal{M}(\mu_i) = Bb^j_i\mu_j \tag{61}$$

Direct use of these results shows that

$$\mathcal{M}(L^j_A\mu_j) = Bb^j_i L^i_A\mu_j \tag{62}$$

Further, it follows directly from

$$dx^i \lrcorner \mu_j = \delta^i_j\mu$$

that

$$d(Bb^j_i L^i_A\mu_j) = \partial_j(Bb^j_i L^i_A)\mu \tag{63}$$

Accordingly, (57) can be written in the equivalent form

$$d\{Bb^j_i L^i_A \mu_j\} - W^\alpha M_{\alpha A}^{\ \ B} \wedge \{Bb^j_i L^i_B \mu_j\} = (\partial L/\partial \Psi^A) B\mu \tag{64}$$

The various terms in the matter field equations are thus seen to arise directly from the minimal replacement construct. Further, (64) shows that these equations are covariant under changes of coordinate covers of M_4. We also know that the field equations (64) are covariant under the local action of P_{10}, and hence under the local action of $L(4, R)$, while W^α are the compensating 1-forms for the local action of $L(4, R)$. Accordingly, we may write (64) in the equivalent form

$$D\{Bb^j_i L^i_A \mu_j\} = (\partial L/\partial \Psi^A) B\mu \tag{65}$$

where D is the $L(4, R)$-gauge covariant exterior derivative

$$DR_A = dR_A - W^\alpha M_{\alpha A}^{\ \ B} \wedge R_B \tag{66}$$

The direct minimal replacement construct is thus seen to induce the $L(4, R)$-gauge connection 1-forms

$$\Gamma^B_A = W^\alpha M_{\alpha A}^{\ \ B} \tag{67}$$

for the matter fields.

If we introduce conjugate basis elements

$$\mu_{ij} = \partial_i \lrcorner \mu_j, \qquad dx^k \wedge \mu_{ij} = \delta^k_i \mu_j - \delta^k_j \mu_i$$

for 2-forms on M_4, a similar analysis shows that (58) and (59) may be written in the equivalent form

$$d\{G^{ij}_k \mu_{ij}\} - W^\alpha C_\alpha{}^m{}_k \wedge \{G^{ij}_m \mu_{ij}\} = -T^m_k Bb^j_m \mu_j + S^j_k \mu_j \tag{68}$$

$$\begin{aligned} d\{G^{ij}_\alpha \mu_{ij}\} - W^\gamma C_\gamma{}^\beta{}_\alpha \wedge \{G^{ij}_\beta \mu_{ij}\} \\ = -(T^m_k Bb^j_m \mu_j) 1_{\alpha r}^{\ k} x^r + \phi^k C_k{}^m{}_\alpha \wedge \{G^{ij}_m \mu_{ij}\} \\ + (Bb^j_k L^k_A \mu_j) M_{\alpha B}^{\ \ A} \Psi^B + S^j_\alpha \mu_j \end{aligned} \tag{69}$$

These equations show that the field equations for the ϕ's and the W's are independent of the choice of coordinate covers of M_4. It is also evident that the left-hand sides of these equations define $L(4, R)$-gauge covariant derivatives in the obvious manners, since these field equations are P_{10}-gauge covariant by constructions.

The terms involving the S's are obviously self-sources of the ϕ and W fields, in view of (50). Noting that T^i_j are the components of the gauge momentum energy complex of the matter fields, it is natural to identify

$$T_k = T^i_k Bb^j_i \mu_j, \qquad 1 \le k \le 4 \tag{70}$$

as the 3-forms of gauge momentum energy; that is,

$$\mathcal{M}(t^i_k \mu_i) = T^i_k B b^j_i \mu_j$$

by (62), where the lower case t's denote the momentum energy complex of the matter fields before minimal replacement. These considerations, in conjunction with (68), show that the ϕ compensating fields are driven by the self-sources and the gauge momentum energy 3-forms of the matter fields. The ϕ-field equations (68) thus have the same general structure as Einstein's field equations; {geometric (compensating) quantities} = {momentum energy quantities}. We will have more to say on this subject in the next section.

The W fields are compensating fields for the local action of $L(4, R)$ and hence the W-field equations (69) should describe spinlike quantities and effects. Indeed, the quantities

$$(T^m_k B b^j_m \mu_j) 1_{\alpha r}^{\ k} x^r = T_k 1_{\alpha r}^{\ k} x^r$$

on the right-hand sides of (69) are four-dimensional moments of the 3-forms of gauge momentum energy of the matter fields. These terms thus represent the "orbital" contributions of the matter fields. Similarly, the terms

$$(B b^j_m L^m_A \mu_j) M_{\alpha B}^{\ A} \Psi^B$$

are easily seen to be the gauge-theoretic analog of spin 3-forms of the matter fields. When we note that

$$1_{\alpha i}^{\ j} = C_{\alpha\ i}^{\ j} = -C_{i\ \alpha}^{\ j} \tag{71}$$

for the standard representation of P_{10} on M_4, the terms on the right-hand side of (69) that involve the ϕ's explicitly are

$$-\phi^k 1_{\alpha k}^{\ m} \wedge \{G^{ij}_k \mu_{ij}\}$$

and may be interpreted as "orbital angular momentum" of the translation compensating fields. These arguments provide a reasonable heuristic basis for accepting the field equations (69) for the $L(4, R)$ compensating fields as also being in the same spirit as the Einstein field equations; albeit four-dimensional "moment" relations.

6. DECOUPLING THE SPIN AND ORBITAL CURRENTS OF MATTER

The fact that the matter fields contribute both a spin current and an orbital current to the field equations for the $L(4, R)$ compensating fields is not all together satisfactory on several grounds. We therefore proceed to eliminate the orbital current from (69).

It is easily seen from (27) that Ω^k can be expressed in terms of θ^α and Σ^k. Accordingly, we may always write

$$V = \bar{V}(x^i, \theta^\alpha_{ij}, \Sigma^k_{ij}, W^\alpha_i, \phi^k_i) \tag{72}$$

with no loss of generality. It then follows directly from (27), (49), and (72) that

$$G^{ij}_k = \partial V/\partial\Omega^k_{ij} = \partial \bar{V}/\partial\Sigma^k_{ij} \tag{73}$$

and hence the field equations (68) are unchanged. On the other hand, the same reasoning gives

$$G^{ij}_\alpha = G^{ij}_k 1_{\alpha m}{}^k x^m + H^{ij}_\alpha \tag{74}$$

where

$$H^{ij}_\alpha = (\partial\bar{V}/\theta^\alpha_{ij})|_\Sigma \tag{75}$$

It is now simply a matter of substituting (74) into (69) and using the Jacobi identity, (12), (13), and (71) several times in order to see that (69) is equivalent to

$$\begin{aligned} &d\{H^{ij}_\alpha\mu_{ij}\} - W^\gamma C_\gamma{}^\beta{}_\alpha \wedge \{H^{ij}_\beta\mu_{ij}\} \\ &\quad = (Bb^j_k L^k_A\mu_j)M_{\alpha B}^A\Psi^B + (S^i_\alpha - 1_{\alpha m}{}^k x^m S^i_k)\mu_i - B^k 1_{\alpha k}{}^m \wedge \{G^{ij}_m\mu_{ij}\} \end{aligned} \tag{76}$$

This simple procedure of using $L(4, R)$ curvature and Cartan torsion, rather than $L(4, R)$ curvature and $T(4)$ curvature, has eliminated the orbital current of the matter fields from the $L(4, R)$ compensating field equations. Particular note should be taken of the fact that the distortion 1-forms B^k have become the "moment arms" in the $\mathbf{G}_m$ contributions to the total spin currents.

There is yet another benefit to be gained from this choice. Now that $\bar{V}$ is a function of $\boldsymbol{\theta}^\alpha$ and $\boldsymbol{\Sigma}^k$, it can be expressed in terms of P_{10}-invariant scalar densities such as α_1 and β_1 given by (36) and (34), respectively. It is then reasonable to expect that $\bar{V}$ will be of the form

$$\bar{V} = U(x^i, \theta^\alpha_{ij}, \Sigma^k_{ij})B \tag{77}$$

In this case, (50) gives

$$S^i_k = UBb^i_k, \qquad S^i_\alpha = S^i_k 1_{\alpha m}{}^k x^m \tag{78}$$

and hence

$$(S^i_\alpha - 1_{\alpha m}{}^k x^m S^i_k)\mu_i \equiv 0 \tag{79}$$

This has the effect of eliminating all of the self-sources from the $L(4, R)$ compensating field equations, for (76) reduces to

$$\begin{aligned} &d\{H^{ij}_\alpha\mu_{ij}\} - W^\gamma C_\gamma{}^\beta{}_\alpha \wedge \{H^{ij}_\beta\mu_{ij}\} \\ &\quad = (Bb^j_k L^k_A\mu_j)M_{\alpha B}^A\Psi^B - B^k 1_{\alpha k}{}^m \wedge \{G^{ij}_m\mu_{ij}\} \end{aligned} \tag{80}$$

where

$$G_k^{ij} = B\, \partial U / \partial \Sigma^k_{ij}, \qquad H_\alpha^{ij} = B(\partial U / \partial \theta^\alpha_{ij})|_\Sigma \tag{81}$$

The field equations of the theory are thus (57) for the matter fields, (58) for the $T(4)$ compensating fields, and either (76) or (80) for the $L(4, R)$ compensating fields. They should be compared and contrasted with the field equations given in the papers cited in the list of references, for there are significant differences.

7. CURRENT CONSERVATION

Current conservation laws are easily obtained directly from (58) and (80) in view of the skew symmetry of $\mathbf{G}_k$ and $\mathbf{H}_\alpha$. We thus have

$$\partial_j\{2 W_i^\alpha C_\alpha{}^m{}_k G_m^{ij} - T_k^m B b_m^j + S_k^j\} = 0 \tag{82}$$

from (58), and

$$\partial_j\{B b_k^j L_A^k M_{\alpha B}^A \Psi^B - B_i^k 1_{\alpha k}^m G_m^{ij} + 2 W_i^\gamma C_\gamma{}^\beta{}_\alpha H_\beta^{ij}\} = 0 \tag{83}$$

We have chosen to write these conservation laws as simple coordinate divergences that have simple and direct interpretations in the underlying space M_4. The reader should carefully note that the forms of these conservation laws can change drastically under gauge transformations since they are not gauge covariant equations. In view of the gauge covariance of the field equations, these conservation laws can be rewritten as gauge covariant conservation laws upon introducing the appropriate gauge covariant exterior derivatives. This only complicates the issue, however, and even hides the essential ingredients, in much the same way that the covariant conservation laws of momentum energy in general relatively conceal some of the essential physics.

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